

JORDAN DERIVATIONS ON LIE IDEALS OF SEMIPRIME χ -RINGS

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ABSTRACT

Let M be a 2-torsion free semiprime Γ -ring satisfying the condition $a\Gamma bSc = aSbrc, \forall a, b, c \in M$ and $r, s \in \Gamma$. Let U be an admissible Lie ideal of M that is, $u\Gamma u \in U, \forall u \in U, r \in \Gamma$ and $U \subseteq Z(M)$, the centre of M . If $d : M \rightarrow M$ is an additive mapping such that d is a Jordan derivation on U of M , then d is a derivation on U .

Keywords: Derivation, Jordan derivation, Lie ideal, admissible Lie ideal, Γ -ring, semiprime Γ -ring

1. Introduction

Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \rightarrow M$ (sending (x, Γ, y) into $x\Gamma y$) such that

- $(x + y)\Gamma z = x\Gamma z + y\Gamma z, x(\Gamma + S)y = x\Gamma y + xSy, x\Gamma(y + z) = x\Gamma y + x\Gamma z$
- $(x\Gamma y)Sz = x\Gamma(ySz), \forall x, y, z \in M$ and $r, s \in \Gamma$

then M is called a Γ -ring. This concept is more general than a ring and was introduced by Barnes [3]. A Γ -ring M is called a *prime Γ -ring* if $\forall a, b \in M, a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$ and M is called *semiprime* if $a\Gamma M\Gamma a = 0$ (with $a \in M$) implies $a = 0$. A Γ -ring M is *2-torsion free* if $2a = 0$ implies $a = 0, \forall a \in M$. For any $x, y \in M$ and $r \in \Gamma$, we denote the commutator $x\Gamma y - y\Gamma x$ by $[x, y]_r$. An additive subgroup $U \subset M$ is said to be a *Lie ideal* of M if whenever $u \in U, m \in M$ and $r \in \Gamma$, then $[u, m]_r \in U$. In the main results of this article we assume that the Lie ideal U verifies $u\Gamma u \in U, \forall u \in U, r \in \Gamma$. A Lie ideal of this type is called a *square closed Lie ideal*. Furthermore, if the Lie ideal U is square closed and $U \not\subseteq Z(M)$, where $Z(M)$ denotes the center of M then U is called an *admissible Lie ideal* of M . Let M be a Γ -ring. An additive mapping $d : M \rightarrow M$ is a *derivation* if $d(a\Gamma b) = d(a)\Gamma b + a\Gamma d(b)$ and a *Jordan derivation* if $d(a\Gamma a) = d(a)\Gamma a + a\Gamma d(a), \forall a, b \in M$ and $r \in \Gamma$. Throughout the article, we use the condition $a\Gamma bSc = aSbrc, \forall a, b, c \in M$ and $r, s \in \Gamma$ and this is denoted by (*). The relationship between usual derivations and Lie ideals of prime rings has been extensively studied in the last forty years. In particular, when this relationship involves the action of the derivations on Lie ideals. In 1984, R. Awtar [2] extended to Lie ideals a

well known result proved by I. N. Herstein [10] which states that 'every Jordan derivation on a 2-torsion free prime ring is a derivation'. In fact, R. Awtar [2] proved that if $U \not\subseteq Z(M)$ is a square closed Lie ideal of a 2-torsion free prime ring R and $d: R \rightarrow R$ is an additive mapping such that $d(u^2) = d(u)u + ud(u), \forall u \in U$, then $d(uv) = d(u)v + ud(v), \forall u, v \in U$. M. Ashraf and N. Rehman [1] studied on Lie ideals and Jordan left derivations of prime rings. They proved that if $d: R \rightarrow R$ is an additive mapping on a 2-torsion free prime ring R satisfying $d(u^2) = 2ud(u), \forall u \in U$, where U is a Lie ideal of R such that $u^2 \in U, \forall u \in U$, then $d(uv) = d(u)v + ud(v), \forall u, v \in U$. A. K. Halder and A. C. Paul [9] extended the results of Y. Ceven [6] in Lie ideals. We have generalized the R. Awtar's [2] result in semiprime Γ -rings.

2. Jordan Derivation on a Lie Ideal

Definition 2.1 Let M be a Γ -ring and U be a Lie ideal of M . An additive mapping $d: M \rightarrow M$ is said to be a Jordan derivation on a Lie ideal U of M if $d(ur u) = d(u)\Gamma u + ur d(u), \forall u \in U$ and $\Gamma \in \Gamma$.

Example 2.1 Let R be a ring of characteristic 2 having a unity element 1.

Let $M = M_{1,2}(R)$ and $N = \left\{ \begin{pmatrix} n & 1 \\ n & 1 \end{pmatrix} : n \in \mathbb{Z}, 2 \nmid n \right\}$.

Then M is a Γ -ring. Let $N = \{(x, x) : x \in R\} \subseteq M$.

Now $\forall (x, x) \in N, (a, b) \in M$ and $\begin{pmatrix} n \\ n \end{pmatrix} \in \Gamma$, we have

$$\begin{aligned} (x, x) \begin{pmatrix} n \\ n \end{pmatrix} (a, b) - (a, b) \begin{pmatrix} n \\ n \end{pmatrix} (x, x) &= (xna - bnx, xnb - anx) \\ &= (xna - 2bnx + bnx, bnx - 2anx + xna) = (xna + bnx, bnx + xna) \in N. \end{aligned}$$

Therefore, N is a Lie ideal of M .

Example 2.2 Let M be a Γ -ring satisfying the condition (*) and let U be a Lie ideal of M . Let $a \in M$ and $\Gamma \in \Gamma$ be fixed elements. Define $d: M \rightarrow M$ by $d(x) = a\Gamma x - x\Gamma a, \forall x \in U$. Now $\forall y \in U$ and $S \in \Gamma$, we have $d(xSy) = a\Gamma xSy - xSy\Gamma a, = (a\Gamma x - x\Gamma a)Sy$ + for every $x, y \in U$ and $S \in \Gamma$. Therefore $d: M \rightarrow M$ is a derivation on U .

Example 2.3 Let M be a Γ -ring and U be a Lie ideal of M . Let $d: M \rightarrow M$ be a derivation on U . Suppose $M_1 = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\Gamma, \Gamma) : \Gamma \in \Gamma\}$. Define addition and multiplication on M_1 as $(x, x) + (y, y) = (x + y, x + y)$ and $(x, x)(\Gamma, \Gamma)(y, y) = (x\Gamma y, x\Gamma y)$. Then M_1 is a Γ_1 -ring. Suppose $U_1 = \{(u, u) : u \in U\}$. Now $(u, u)(\Gamma, \Gamma)(x, x) - (x, x)(\Gamma, \Gamma)(u, u) = (u\Gamma x, u\Gamma x) - (x\Gamma u, x\Gamma u) = (u\Gamma x - x\Gamma u, u\Gamma x - x\Gamma u) \in U_1$ for $u\Gamma x - x\Gamma u \in U$. Hence U_1 is a Lie ideal of M_1 . If we define a mapping

$D: M_1 \rightarrow M_1$ on U_1 by $D((u, u)) = (d(u), d(u))$. Then it is clear that D is a Jordan derivation on U_1 which is not a derivation on U_1 .

Lemma 2.1 *Let M be a Γ -ring and U be a Lie ideal of M such that $u\Gamma u \in U, \forall u \in U$ and $r \in \Gamma$. If d is a Jordan derivation on U of M , then $\forall a, b, c \in U$ and $r, s \in \Gamma$, the following statements hold:*

- (i) $d(arb + bra) = d(a)rb + d(b)ra + ar d(b) + br d(a)$.
- (ii) $d(arbsa + asbra) = d(a)rbsa + d(a)sbra + ar d(b)sa + asd(b)ra + arbsd(a) + asbrd(a)$ In particular, if M is 2-torsion free and satisfies the condition (*), then
- (iii) $d(arbsa) = d(a)rbsa + ar d(b)sa + arbsd(a)$.
- (iv) $d(arbsc + crbsa) = d(a)rbsc + d(c)rbsa + ar d(b)sc + cr d(b)sa + arbsd(c) + crbsd(a)$.

Proof. Since U is a Lie ideal satisfying the condition $a\Gamma a \in U, \forall a \in U, r \in \Gamma$. For $a, b \in U, r \in \Gamma, (arb + bra) = (a + b)r(a + b) - (ara + brb)$ and so $(arb + bra) \in U$.

Also, $[a, b]_r = arb - bra \in U$ and it follows that $2arb \in U$.

Hence $4arbsc = 2(2arb)sc \in U, \forall a, b, c \in U, r, s \in \Gamma$.

Thus

$$\begin{aligned} d(arb + bra) &= d((a + b)r(a + b) - (ara + brb)) \\ &= d(a + b)r(a + b) + (a + b)r d(a + b) - d(a)ra - ar d(a) - d(b)rb - br d(b) \\ &= d(a)ra + d(a)rb + d(b)ra + d(b)rb + ar d(a) + ar d(b) + br d(a) + br d(b) - \\ &\quad d(a)ra - ar d(a) - d(b)rb - br d(b) = d(a)rb + ar d(b) + d(b)ra + br d(a). \end{aligned}$$

Replacing $asb + bsa$ for b in (i) we get

$$\begin{aligned} d(ar(asb + bsa) + (asb + bsa)ra) \\ = d(a)r(asb + bsa) + ar d(asb + bsa) + d(asb + bsa)ra + (asb + bsa)r d(a). \end{aligned}$$

This implies that

$$\begin{aligned} d(ara)sb + (ara)sd(b) + d(b)s(ara) + bsd(ara) + d(arbsa + asbra) \\ = d(a)rasb + d(a)rbsa + ar d(a)sb + ar asd(b) + ar d(b)sa + arbsd(a) \\ + d(a)sbra + asd(b)ra + d(b)sara + bsd(a)ra + asbrd(a) + bsard(a). \end{aligned}$$

This implies that

$$\begin{aligned} d(a)r asb + ar d(a)sb + ar asd(b) + d(b)sa\Gamma a + bsd(a)\Gamma a + bsard(a) + d(arbsa + asbra) \\ = d(a)r asb + d(a)rbsa + ar d(a)sb + ar asd(b) + ar d(b)sa + arbsd(a) + d(a)sbra \\ + asd(b)\Gamma a + d(b)sa\Gamma a + bsd(a)\Gamma a + asbrd(a) + bsard(a). \end{aligned}$$

Now canceling the like terms from both sides we get the required result. Using the condition (*) and since M is 2-torsion free, (iii) follows from (ii). And finally (iv) is obtained by replacing $a + c$ for a in (iii).

Lemma 2.2 *Let U be a Lie ideal of a 2-torsion free Γ -ring M satisfying the condition(*) then $T(U) = \{x \in M : [x, M]_{\Gamma} \subseteq U\}$ is both a subring and a Lie ideal of M such that $U \subseteq T(U)$.*

Proof. We have U is a Lie ideal of M , so $[U, M]_{\Gamma} \subseteq U$. Thus $U \subseteq T(U)$. Also we have $[T(U), M]_{\Gamma} \subseteq U \subseteq T(U)$. Hence $T(U)$ is a Lie ideal of M . Now suppose that $x, y \in T(U)$ then $[x, m]_{\Gamma} \in U$ and $[y, m]_{\Gamma} \in U, \forall m \in M$ and $r \in \Gamma$. Now $[xry, m]_{\Gamma} = xr[y, m]_{\Gamma} + [x, m]_{\Gamma} r y \in U$. Therefore, $[xry, m]_{\Gamma} \in U, \forall x, y \in T(U), m \in M$ and $r, s \in \Gamma$. Hence $xry \in T(U)$.

Lemma 2.3 *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free semiprime Γ -ring M satisfying the condition (*), then there exists a nonzero ideal $K = M\Gamma[U, U]_{\Gamma}M$ of M generated by $[U, U]_{\Gamma}$ such that $[K, M]_{\Gamma} \subseteq U$.*

Proof. First we prove that if $[U, U]_{\Gamma} = 0$ then $U \subseteq Z(M)$, so let $[U, U]_{\Gamma} = 0$ for $u \in U$ and $r \in \Gamma$, we have $[u, [u, x]_{\Gamma}]_{\Gamma} = 0, \forall x \in M$. For all $z \in M$ and $s \in \Gamma$, we replace x by xSz in $[u, [u, x]_{\Gamma}]_{\Gamma} = 0$ and obtain

$$\begin{aligned} 0 &= [u, [u, xSz]_{\Gamma}]_{\Gamma} \\ &= [u, xS[u, z]_{\Gamma} + [u, x]_{\Gamma} Sz]_{\Gamma} \\ &= [u, xS[u, z]_{\Gamma}]_{\Gamma} + [u, [u, x]_{\Gamma} Sz]_{\Gamma} \\ &= xS[u, [u, z]_{\Gamma}]_{\Gamma} + [u, x]_{\Gamma} S[u, z]_{\Gamma} + [u, [u, x]_{\Gamma}]_{\Gamma} Sz + [u, x]_{\Gamma} S[u, z]_{\Gamma} \\ &= 2[u, x]_{\Gamma} S[u, z]_{\Gamma} \end{aligned}$$

By the 2-torsion freeness of M , we obtain $[u, x]_{\Gamma} S[u, z]_{\Gamma} = 0$. Now replacing z by $z\chi$, we obtain

$$\begin{aligned} 0 &= [u, x]_{\Gamma} S[u, z\chi]_{\Gamma} \\ &= [u, x]_{\Gamma} Sz\chi[u, x]_{\Gamma} + [u, x]_{\Gamma} S[u, z]_{\Gamma} \chi \\ &= [u, x]_{\Gamma} Sz\chi[u, x]_{\Gamma} \end{aligned}$$

That is, $[u, x]_{\Gamma} S M \chi [u, x]_{\Gamma} = 0$. Since M is semiprime, $[u, x]_{\Gamma} = 0$. This implies that $u \in Z(M)$ and therefore, $U \subseteq Z(M)$ is a contradiction. So let $[U, U]_{\Gamma} \neq 0$. Then $K = M\Gamma[U, U]_{\Gamma}M$ is a nonzero ideal of M generated by $[U, U]_{\Gamma}$. Let $x, y \in U, m \in M$ and $r, s \in \Gamma$, we have $[x, ySm]_{\Gamma}, y, [x, m]_{\Gamma} \in U \subseteq T(U)$.

Hence $[x, y]_{\Gamma} Sm = [x, ySm]_{\Gamma} - yS[x, m]_{\Gamma} \in T(U)$.

Also we can show that, $mS[x, y]_{\Gamma} \in T(U)$ and therefore, we obtain $[[U, U]_{\Gamma}, M]_{\Gamma} \subseteq U$. That is, $[[[[x, y]_{\Gamma}, m]_{\Gamma}, s]_{\Gamma}, t]_{\Gamma} \in U, \forall m, s, t \in M$ and $r \in \Gamma$.

Hence $[x, y]_{\Gamma} \Gamma m \Gamma s - m \Gamma [x, y]_{\Gamma} \Gamma s + [s, m]_{\Gamma} \Gamma [x, y]_{\Gamma} - [s \Gamma [x, y]_{\Gamma}, m]_{\Gamma} \Gamma t]_{\Gamma} \in T(U)$.

Since $[x, y]_{\Gamma} \Gamma m \Gamma s, s \Gamma [x, y]_{\Gamma}, [s, m]_{\Gamma} \Gamma [x, y]_{\Gamma} \in T(U)$.

Thus we have, $[m \Gamma [x, y]_{\Gamma} \Gamma s, t]_{\Gamma} \in U, \forall m, s, t \in M$ and $r \in \Gamma$. Hence $[K, M]_{\Gamma} \subseteq U$.

Lemma 2.4 *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free semiprime Γ -ring M satisfying the condition (*) and $a \in U$. If $a \Gamma U S a = \{0\}, \forall r, s \in \Gamma$ then $a \Gamma a = 0$ and there exists a nonzero ideal $K = M \Gamma [U, U]_{\Gamma} \Gamma M$ of M generated by $[U, U]_{\Gamma}$ such that $[K, M]_{\Gamma} \subseteq U$ and $K \Gamma a = a \Gamma K = \{0\}$.*

Proof. If $a \Gamma U S a = \{0\}, \forall r, s \in \Gamma$, then $a \Gamma [a, a \Gamma m]_{\Gamma} S a = 0, \forall m \in M$ and $u \in \Gamma$. Thus, by using (*)

$$\begin{aligned} 0 &= a \Gamma (a \Gamma a \Gamma u m - a \Gamma u m \Gamma a) S a \\ &= a \Gamma a \Gamma a \Gamma u m S a - a \Gamma a \Gamma u m \Gamma a S a \\ &= a \Gamma a \Gamma u a \Gamma m - a \Gamma a \Gamma u m S a \Gamma a. \end{aligned}$$

Since $a \Gamma a \Gamma u a = 0$, we have $(a \Gamma a) \Gamma u m S (a \Gamma a) = 0$. Since M is semiprime, $a \Gamma a = 0$. Now we obtain $a \Gamma [k \Gamma a, m]_{\Gamma} \Gamma u S a = 0, \forall k \in K, m \in M, u \in U$ and $r, s, \sim, \in \Gamma$. Therefore, by using (*) and $a \Gamma U S a = \{0\}$.

$$\begin{aligned} 0 &= a \Gamma (k \Gamma a \sim m - m \sim k \Gamma a) \Gamma u S a \\ &= a \Gamma k \Gamma a \sim m \Gamma u S a - a \Gamma m \sim k \Gamma a \Gamma u S a \\ &= a \Gamma k \Gamma a \sim m S u \Gamma a. \end{aligned}$$

So we obtain, $a \Gamma k \Gamma a \sim m S [k, a]_{\Gamma} \Gamma a = 0$. This implies that $a \Gamma k \Gamma a \sim m S (k \Gamma a - a \Gamma k) \Gamma a = 0$ and hence $a \Gamma k \Gamma a \sim m S k \Gamma a \Gamma a - a \Gamma k \Gamma a \sim m S a \Gamma k \Gamma a = 0$. By using (*) and $a \Gamma a = 0$, we obtain, $(a \Gamma k \Gamma a) \sim m S (a \Gamma k \Gamma a) = 0$. Since M is semiprime, $a \Gamma k \Gamma a = 0$. Thus we find that $(a \Gamma k) \Gamma M S (a \Gamma k) = 0$. Hence $a \Gamma k = 0, \forall k \in K$, that is $a \Gamma K = \{0\}$. Similarly we obtain $K \Gamma a = \{0\}$.

Lemma 2.5 *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free semiprime Γ -ring M satisfying the condition (*). Let $a, b \in U$ and $r, s \in \Gamma$.*

- (i) If $a \Gamma U S a = \{0\}$, then $a = 0$.
- (ii) If $a \Gamma U = \{0\}$ (or $U \Gamma a = \{0\}$), then $a = 0$.
- (iii) If $u \Gamma u \in U, \forall u \in U$ and $a \Gamma U S b = \{0\}$ then $a \Gamma b = 0$ and $b \Gamma a = 0, \forall r \in \Gamma$.

Proof. (i) By Lemma 2.4, we have $K\Gamma a = M\Gamma[U, U]_{\Gamma} \Gamma M\Gamma a = \{0\}$ and $a\Gamma a = 0 \forall \Gamma \in \Gamma$.

Therefore, $\forall x, y \in M$ and $\Gamma, S \in \Gamma$, we obtain:

$$\begin{aligned} 0 &= [[a, x]_{\Gamma}, a]_{\Gamma} S\Gamma a \\ &= [a\Gamma x - x\Gamma a, a]_{\Gamma} S\Gamma a \\ &= a\Gamma [x, a]_{\Gamma} S\Gamma a - [x, a]_{\Gamma} \Gamma a S\Gamma a \\ &= a\Gamma x\Gamma a S\Gamma a - a\Gamma a\Gamma x S\Gamma a - x\Gamma a\Gamma a S\Gamma a + a\Gamma x\Gamma a S\Gamma a \\ &= a\Gamma x\Gamma a S\Gamma a + a\Gamma x\Gamma a S\Gamma a \\ &= 2a\Gamma x\Gamma a S\Gamma a. \end{aligned}$$

By the 2-torsion freeness of M , we have $a\Gamma x\Gamma a S\Gamma a = 0$. Thus we obtain, $a\Gamma x\Gamma a S\Gamma a U\Gamma x\Gamma a = 0$. By using (*), we have $(a\Gamma x\Gamma a)S\Gamma U(a\Gamma x\Gamma a) = 0$. This implies that $(a\Gamma x\Gamma a)S\Gamma M\Gamma U(a\Gamma x\Gamma a) = 0$. Since M is semiprime $a\Gamma x\Gamma a = 0, \forall x \in M$ and $\Gamma, x \in \Gamma$. Again using the semiprimeness of M , we obtain $a = 0$.

(ii) If $a\Gamma U = \{0\}$, then $a\Gamma U S\Gamma a = \{0\}, \forall S \in \Gamma$, therefore by (i), we obtain $a = 0$. Similarly, if $U\Gamma a = \{0\}$, then $a = 0$.

(iii) If $a\Gamma U S\Gamma b = \{0\}$, then we have $(b\Gamma x\Gamma a)\Gamma U S\Gamma (b\Gamma x\Gamma a) = \{0\}$ and hence by (i), $b\Gamma x\Gamma a = 0, \forall x \in \Gamma$. Also $(a\Gamma x\Gamma b)\Gamma U S\Gamma (a\Gamma x\Gamma b) = \{0\}$ if $a\Gamma U S\Gamma b = \{0\}$ and hence $a\Gamma x\Gamma b = 0$.

Definition 2.2 Let M be a 2-torsion free semiprime Γ -ring satisfying the condition (*) and U be a Lie ideal of M . Let D be a Jordan derivation on U of M . Then $\forall a, b \in U$ and $\Gamma \in \Gamma$, we define $G_{\Gamma}(a, b) = d(a\Gamma b) - d(a)\Gamma b - a\Gamma d(b)$.

Lemma 2.6 Let M be 2-torsion free semiprime Γ -ring satisfying the condition (*) and U be a square closed Lie ideal of M . Let d be a Jordan derivation on U of M . Then $\forall a, b, c \in U$ and $\Gamma, S \in \Gamma$, the following statements hold.

- (i) $G_{\Gamma}(a, b) + G_{\Gamma}(b, a) = 0$;
- (ii) $G_{\Gamma}(a + b, c) = G_{\Gamma}(a, c) + G_{\Gamma}(b, c)$;
- (iii) $G_{\Gamma}(a, b + c) = G_{\Gamma}(a, b) + G_{\Gamma}(a, c)$;
- (iv) $G_{\Gamma+S}(a, b) = G_{\Gamma}(a, b) + G_S(a, b)$.

Remark 2.1 d is a derivation on U of M if and only if $G_{\Gamma}(a, b) = 0, \forall a, b \in U$ and $\Gamma \in \Gamma$.

Remark 2.2 If U is a commutative Lie ideal of M , then by Lemma 2.3 $U \subseteq Z(M)$. In this situation, we have seen that $G_{\Gamma}(a, b) = 0, \forall a, b \in U$ and $\Gamma \in \Gamma$ which is obtained from Lemma 2.1(i). So, we proved our main result for admissible Lie ideal of M .

Lemma 2.7 Let M be 2-torsion free Γ -ring satisfying the condition (*) and U be a square closed Lie ideal of M . If d is a Jordan derivation on U of M . Then

$$(i)G_{\Gamma}(a,b)Sux[a,b]_{\Gamma}+[a,b]_{\Gamma}SuxG_{\Gamma}(a,b)=0,\forall a,b,u\in U \text{ and } r,s\in\Gamma$$

$$(ii)G_{\Gamma}(a,b)rur[a,b]_{\Gamma}+[a,b]_{\Gamma}rurG_{\Gamma}(a,b)=0,\forall a,b,u\in U \text{ and } r,s\in\Gamma$$

$$(iii)G_{\Gamma}(a,b)Sus[a,b]_{\Gamma}+[a,b]_{\Gamma}SusG_{\Gamma}(a,b)=0,\forall a,b,u\in U \text{ and } r,s\in\Gamma.$$

Proof. (i) For any $a,b,u\in U$ and $r,s,\chi\in\Gamma$, we have by using Lemma 2.1(iv)

$$\begin{aligned} W &= d(4arbsuxbr a + 4br asuxar b) \\ &= d((2ar b)Sux(2br a) + (2br a)Sux(2ar b)) \\ &= 4d(ar b)Suxbr a + 4ar bsd(u)\chi br a + 4ar bSuxd(br a) \\ &\quad + 4d(br a)Suxar b + 4br asd(u)\chi ar b + 4br asuxd(ar b). \end{aligned}$$

On the other hand by using Lemma 2.1(iii)

$$\begin{aligned} W &= d(ar(4bsuxb)ra + br(4asuxa)rb) \\ &= d(ar(4bsuxb)ra) + d(br(4asuxa)rb) \\ &= 4d(a)rbSuxbr a + 4ard(bsuxb)ra + 4arbsuxbr d(a) + 4d(b)rasuxar b \\ &\quad + 4br d(asuxa)rb + 4br asuxar d(b) \\ &= 4(d(a)rbSuxbr a + ard(b)Suxbr a + ar bsd(u)\chi br a + ar bSuxd(b)ra \\ &\quad + ar bSuxbr d(a) + d(b)rasuxar b + br d(a)Suxar b + br asd(u)\chi ar b \\ &\quad + br asuxd(a)rb + br asuxar d(b)). \end{aligned}$$

Equating two expressions for W and canceling the like terms from both sides, we get

$$\begin{aligned} &4(d(ar b)Suxbr a + ar bSuxd(br a) + d(br a)Suxar b + br asuxd(ar b)) \\ &= 4(d(a)rbSuxbr a + ard(b)Suxbr a + ar bSuxd(b)ra + ar bSuxbr d(a) \\ &\quad + d(b)rasuxar b + br d(a)Suxar b + br asuxd(a)rb + br asuxar d(b)). \end{aligned}$$

This gives

$$\begin{aligned} &4(d(ar b)Suxbr a - d(a)rbSuxbr a - ard(b)Suxbr a + d(br a)Suxar b \\ &\quad - d(b)rasuxar b - br d(a)Suxar b + ar bSuxd(br a) - ar bSuxd(b)ra \\ &\quad - ar bSuxbr d(a) + br asuxd(ar b) - br asuxd(a)rb - br asuxar d(b)) = 0. \end{aligned}$$

This implies that

$$\begin{aligned} &4((d(ar b) - d(a)rb - ard(b))Suxbr a + (d(br a) - d(b)ra - br d(a))Suxar b + \\ &\quad ar bSux(d(br a) - d(b)ra - br d(a)) + br asux(d(ar b) - d(a)rb - ard(b))) = 0. \end{aligned}$$

Now using the Definition 2.2 and 2-torsion freeness of M , we obtain

$$G_{\Gamma}(a,b)Suxbr a + G_{\Gamma}(b,a)Suxar b + ar bSuxG_{\Gamma}(b,a) + br asuxG_{\Gamma}(a,b) = 0.$$

This implies that

$$G_{\Gamma}(a,b)Sux[a,b]_{\Gamma}+[a,b]_{\Gamma}SuxG_{\Gamma}(a,b)=0,\forall a,b,u\in U,r,s,\chi\in\Gamma.$$

If we consider

$$W = d(4a\Gamma b\Gamma u\Gamma r\Gamma b\Gamma r\Gamma a + 4b\Gamma a\Gamma u\Gamma r\Gamma a\Gamma r\Gamma b) \text{ and } W = d(4a\Gamma r\Gamma b\Gamma s\Gamma u\Gamma s\Gamma b\Gamma r\Gamma a + 4b\Gamma r\Gamma a\Gamma s\Gamma u\Gamma s\Gamma a\Gamma r\Gamma b)$$

for (ii) and (iii) respectively and proceeding in the same way as in the proof of (i) by the similar arguments, we get (ii) and (iii).

Lemma 2.8 *Let U be an admissible Lie ideal of a 2-torsion free semiprime Γ -ring M and let $a, b \in U$. If $a\Gamma u\Gamma s\Gamma b + b\Gamma r\Gamma u\Gamma s\Gamma a = 0, \forall u \in U, r, s \in \Gamma$, then $a\Gamma u\Gamma s\Gamma b = 0 = b\Gamma r\Gamma u\Gamma s\Gamma a$.*

Proof. Let $x \in U$ and $\chi \in \Gamma$ be any elements. Using the relation $a\Gamma u\Gamma s\Gamma b + b\Gamma r\Gamma u\Gamma s\Gamma a = 0, \forall u \in U, r, s \in \Gamma$ repeatedly, we get

$$\begin{aligned} 4(a\Gamma u\Gamma s\Gamma b)\chi\chi\chi(a\Gamma u\Gamma s\Gamma b) &= -4(b\Gamma r\Gamma u\Gamma s\Gamma a)\chi\chi\chi(a\Gamma u\Gamma s\Gamma b) \\ &= -(b\Gamma(4u\Gamma s\Gamma a\chi\chi)\chi a)\Gamma u\Gamma s\Gamma b \\ &= (a\Gamma(4u\Gamma s\Gamma a\chi\chi)\chi b)\Gamma u\Gamma s\Gamma b \\ &= a\Gamma u\Gamma s\Gamma(4a\chi\chi\chi b)\Gamma u\Gamma s\Gamma b \\ &= -a\Gamma u\Gamma s\Gamma(4b\chi\chi\chi a)\Gamma u\Gamma s\Gamma b \\ &= -4(a\Gamma u\Gamma s\Gamma b)\chi\chi\chi(a\Gamma u\Gamma s\Gamma b). \end{aligned}$$

This implies, $8((a\Gamma u\Gamma s\Gamma b)\chi\chi\chi(a\Gamma u\Gamma s\Gamma b)) = 0$. Since M is 2-torsion free, $(a\Gamma u\Gamma s\Gamma b)\chi\chi\chi(a\Gamma u\Gamma s\Gamma b) = 0$. Therefore, $(a\Gamma u\Gamma s\Gamma b)\chi U\chi(a\Gamma u\Gamma s\Gamma b) = 0$. Thus by Lemma 2.5, we get $a\Gamma u\Gamma s\Gamma b = 0$. Similarly, it can be shown that $b\Gamma r\Gamma u\Gamma s\Gamma a = 0$.

Corollary 2.1 *Let M be a 2-torsion free semiprime Γ -ring satisfying the condition (*) and U be an admissible Lie ideal of M . If d is a Jordan derivation on U of M . Then $\forall a, b, u \in U, r, s, \chi \in \Gamma$*

$$\begin{aligned} (i)G_r(a, b)Su\chi[a, b]_r &= 0; (ii)[a, b]_r Su\chi G_r(a, b) = 0; (iii)G_r(a, b)\Gamma u\Gamma r[a, b]_r = 0; \\ (iv)[a, b]_r \Gamma u\Gamma r G_r(a, b) &= 0; (v)G_r(a, b)SuS[a, b]_r = 0; (vi)[a, b]_r SuS G_r(a, b) = 0. \end{aligned}$$

Proof. Applying the result of Lemma 2.8 in that of Lemma 2.7, we obtain these results.

Lemma 2.9 *Let M be a 2-torsion free semiprime Γ -ring satisfying the condition (*) and U be an admissible Lie ideal of M . Let d be a Jordan derivation on U of M . Then $\forall a, b, u \in U, r, s, \chi \in \Gamma$*

$$\begin{aligned} (i)G_r(a, b)SuS[x, y]_r &= 0; (ii)[x, y]_r SuS G_r(a, b) = 0 \\ (iii)G_r(a, b)SuS[x, y]_\chi &= 0; (iv)[x, y]_\chi SuS G_r(a, b) = 0 \end{aligned}$$

Proof. (i) If we substitute $a + x$ for a in the Corollary 2.1 (v), we get

$$G_r(a + x, b)SuS[a + x, b]_r = 0$$

This implies

$$G_r(a, b)SuS[a, b]_r + G_r(a, b)SuS[x, b]_r + G_r(x, b)SuS[a, b]_r + G_r(x, b)SuS[x, b]_r = 0.$$

Thus by using Corollary 2.1 (v), we have

$$G_r(a, b)SuS[x, b]_r + G_r(x, b)SuS[a, b]_r = 0.$$

Thus, we obtain

$$\begin{aligned} & (G_r(a, b)SuS[x, b]_r)SuS(G_r(a, b)SuS[x, b]_r) \\ &= -G_r(a, b)SuS[x, b]_r SuSG_r(x, b)SuS[a, b]_r = 0. \end{aligned}$$

Hence, by Lemma 2.5(i), we get

$$G_r(a, b)SuS[x, b]_r = 0$$

Similarly, by replacing $b + y$ for b in this result, we get

$$G_r(a, b)SuS[x, y]_r = 0$$

(ii) Proceeding in the same way as described above by the similar replacements successively in Corollary 2.1 (vi), we obtain

$$[x, y]_x SuSG_r(a, b) = 0, \forall a, b, x, y, u \in U, r, s \in \Gamma$$

(iii) Replacing $r + x$ for r in (i), we get

$$G_{r+x}(a, b)SuS[x, y]_{r+x} = 0$$

This implies

$$(G_r(a, b) + G_x(a, b))SuS([x, y]_r + [x, y]_x) = 0$$

Therefore

$$G_r(a, b)SuS[x, y]_r + G_r(a, b)SuS[x, y]_x + G_x(a, b)SuS[x, y]_r + G_x(a, b)SuS[x, y]_x = 0.$$

Thus by using Corollary 2.1 (vi), we get

$$G_r(a, b)SuS[x, y]_x + G_x(a, b)SuS[x, y]_r = 0.$$

Thus, we obtain

$$\begin{aligned} & (G_r(a, b)SuS[x, y]_x)SuS(G_r(a, b)SuS[x, y]_x) \\ &= -G_r(a, b)SuS[x, y]_x SuSG_x(a, b)SuS[x, y]_r = 0. \end{aligned}$$

Hence, by Lemma 2.5(i), we obtain $G_r(a, b)SuS[x, y]_x = 0$.

(iv) As in the proof of (iii), the similar replacement in (ii) produces (iv).

Lemma 2.10 ([8], Lemma 3.6.1) *Every semiprime Γ -ring contains no nonzero nilpotent ideal.*

Corollary 2.2 ([8], Corollary 3.6.2) *Semiprime Γ -ring has no nonzero nilpotent element.*

Lemma 2.11 ([8], Lemma 3.6.2) *The center of a semiprime Γ -ring does not contain any nonzero nilpotent element.*

Theorem 2.1 *Let M be a 2-torsion free semiprime Γ -ring satisfying the condition(*) and U be an admissible Lie ideal of M . If $d : M \rightarrow M$ is a Jordan derivation on U , then d is a derivation on U of M .*

Proof. Let $a, b, y, u \in U$ and $r, s \in \Gamma$. Then by Lemma 2.9 (iii), we get

$$\begin{aligned} 2[G_r(a, b), y]_s s u s [G_r(a, b), y]_s &= 2(G_r(a, b) s y - y s G_r(a, b)) s u s [G_r(a, b), y]_s \\ &= G_r(a, b) s (2y s u) s [G_r(a, b), y]_s - 2y s G_r(a, b) s u s [G_r(a, b), y]_s = 0. \end{aligned}$$

Since $2y s u \in U$ and $G_r(a, b) \in M, \forall a, b, y, u \in U$ and $r, s \in \Gamma$. Hence, by Lemma 2.5(i), $[G_r(a, b), y]_s = 0$, where $G_r(a, b) \in M, \forall a, b, y \in U$ and $r, s \in \Gamma$. Therefore, $G_r(a, b) \in Z(M)$, the center of M . Now, let $x \in U$ and $\chi, u \in \Gamma$. Then, we have $G_r(a, b) \chi [x, y]_r u u G_r(a, b) \chi [x, y]_r = 0$, by using Lemma 2.9(ii).

Thus by using Lemma 2.5(i)

$$G_r(a, b) \chi [x, y]_r = 0. \quad (1)$$

Similarly, we get $[x, y]_r \chi G_r(a, b) u u [x, y]_r \chi G_r(a, b) = 0$, by using Lemma 2.9(i).

Now by Lemma 2.5(i), we get

$$[x, y]_r \chi G_r(a, b) = 0. \quad (2)$$

Similarly, by Lemma 2.9(iv), we have $G_r(a, b) \chi [x, y]_s u u G_r(a, b) \chi [x, y]_s = 0$. By Lemma 2.5(i), it follows that

$$G_r(a, b) \chi [x, y]_s = 0. \quad (3)$$

Also, by Lemma 2.9(iii), we have $[x, y]_s \chi G_r(a, b) u u [x, y]_s \chi G_r(a, b) = 0$. Hence by Lemma 2.5(i), we get

$$[x, y]_s \chi G_r(a, b) = 0. \quad (4)$$

Thus, we have

$$\begin{aligned} 2G_r(a, b) \chi G_r(a, b) &= G_r(a, b) \chi (G_r(a, b) + G_r(a, b)) \\ &= G_r(a, b) \chi (G_r(a, b) - G_r(b, a)) \\ &= G_r(a, b) \chi (d(arb) - d(a)r b - ar d(b) - d(bra) + d(b)ra + br d(a)) \\ &= G_r(a, b) \chi (d(arb - bra) + (br d(a) - d(a)r b) + (d(b)ra - ar d(b))) \\ &= G_r(a, b) \chi (d([a, b]_r) + [b, d(a)]_r + [d(b), a]_r) \\ &= G_r(a, b) \chi d([a, b]_r) - G_r(a, b) \chi [d(a), b]_r - G_r(a, b) \chi [a, d(b)]_r. \end{aligned}$$

Since $d(a), d(b) \in M$, so by using Lemma 2.9(i), we get

$$G_r(a, b) \chi [d(a), b]_r = G_r(a, b) \chi [a, d(b)]_r = 0.$$

Thus

$$2G_r(a, b) \times G_r(a, b) = G_r(a, b) \times d([a, b]_r). \quad (5)$$

Adding (3) and (4), we obtain $G_r(a, b) \times [x, y]_s + [x, y]_s \times G_r(a, b) = 0$. Then by Lemma 2.1 (i) with the use of (3), we have

$$\begin{aligned} 0 &= d(G_r(a, b) \times [x, y]_s + [x, y]_s \times G_r(a, b)) \\ &= d(G_r(a, b)) \times [x, y]_s + d([x, y]_s) \times G_r(a, b) + G_r(a, b) \times d([x, y]_s) + [x, y]_s \times d(G_r(a, b)) \\ &= d(G_r(a, b)) \times [x, y]_s + 2G_r(a, b) \times d([x, y]_s) + [x, y]_s \times d(G_r(a, b)). \end{aligned}$$

Since $G_r(a, b) \in Z(M)$ and therefore $d([x, y]_s) \times G_r(a, b) = G_r(a, b) \times d([x, y]_s)$.

Hence, we get

$$2G_r(a, b) \times d([x, y]_s) = -d(G_r(a, b)) \times [x, y]_s - [x, y]_s \times d(G_r(a, b)). \quad (6)$$

Then from (5) and (6) we have

$$\begin{aligned} 4G_r(a, b) \times G_r(a, b) &= 2G_r(a, b) \times d([a, b]_r) \\ &= -d(G_r(a, b)) \times [a, b]_r - [a, b]_r \times d(G_r(a, b)). \end{aligned}$$

Thus we obtain

$$\begin{aligned} 4G_r(a, b) \times G_r(a, b) \times G_r(a, b) \times G_r(a, b) &= \\ -G_r(a, b) \times d(G_r(a, b)) \times [a, b]_r \times G_r(a, b) &- G_r(a, b) \times [a, b]_r \times d(G_r(a, b)) \times G_r(a, b). \end{aligned}$$

Since by (1) and (2), $G_r(a, b) \times [a, b]_r \times d(G_r(a, b)) \times G_r(a, b) = 0$ and

$$G_r(a, b) \times d(G_r(a, b)) \times [a, b]_r \times G_r(a, b) = 0.$$

So, we get

$$4G_r(a, b) \times G_r(a, b) \times G_r(a, b) \times G_r(a, b) = 0.$$

Therefore, $4(G_r(a, b) \times)^3 G_r(a, b) = 0$. Since M is 2-torsion free, so we have

$$(G_r(a, b) \times)^3 G_r(a, b) = 0$$

But, it follows that $G_r(a, b)$ is a nilpotent element of the Γ -ring M . Since by Lemma 2.11, the center of a semiprime Γ -ring does not contain any nonzero nilpotent element, so we get $G_r(a, b) = 0, \forall a, b \in U$ and $r \in \Gamma$. It means that, d is a derivation on U of M .

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